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Commutative polynomial semigroups on a segment

by

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1. Introduction

A commutative semigroup of mappings of a set X is a family of mappings $X \to X$ which is a commutative semigroup under composition of functions. A commutative polynomial semigroup of mappings of a subset X of the real line R (shortly: an X-cps) is a commutative semigroup of mappings $X \to X$, all elements of which are restrictions to X of (real) polynomials on R. Such a semigroup S is called <u>maximal</u> if every continuous map $g : X \to X$ which commutes with all $f \in S$ itself belongs to S, and <u>entire</u> if it contains (restrictions to X of) polynomials of every non-negative degree.

If S_i is a semigroup of continuous maps $X_i \to X_i$ ($i = 1,2$), and if τ is a homeomorphism of X_1 onto X_2 such that $S_2 = \{\tau \circ f \circ \tau^{-1} \mid f \in S_1\}$, then S_1 and S_2 are called <u>equivalent</u> (by means of τ). In that case the transformation $f \to \tau \circ f \circ \tau^{-1}$ is an isomorphism of the abstract semigroup S_1 onto the abstract semigroup S_2.

In this note we determine, up to equivalence, all entire I-cps, where I is the closed unit segment $[0,1]$. Moreover, we establish which of these I-cps are maximal and which not.

2. Commutative polynomial semigroups of mappings $R \to R$

It follows from results of J.F. Ritt [6,7] and of H.D. Block and H.P. Thielman [5] that the every entire R-cps is equivalent by means of a linear transformation to one of the following three semigroups of polynomials:

(i) the semigroup P, consisting of the maps $P_0, P_1 P_2, \ldots$ with

$$P_n(x) = x^n \quad ;$$

(ii) the semigroup P^*, consisting of all P_n, $n \geqslant 1$, and the map P_o^* such that

$$P_o^*(x) = 0 \quad \text{for all } x \;;$$

(iii) the semigroup T of all Chebyshev polynomials $T_o, T_1, T_2, \ldots$, where

$$T_n(x) = \cos\,(n.\ \mathrm{arc}\cos x).$$

The first two semigroups are not maximal; in fact:

Lemma 1. There exists a unique maximal commutative semigroup $\bar{P}$ ($\overline{P^*}$) of continuous maps $R \to R$ containing P (P^*, respectively). The semigroup $\bar{P}$ ($\overline{P^*}$) consists of the following maps: all maps $x \to |x|^{\varepsilon}$, ε a real number; all maps $x \to |x|^{\varepsilon}$. sign x , ε a real number; and all maps in P (in P^*, respectively).

Proof.

It is immediately verified that $\bar{P}$ and $\overline{P^*}$ are commutative semigroups. In order to show their maximality, and the fact that they are the only maximal semigroups containing $\bar{P}$ or $\overline{P^*}$, we proceed as follows.

Let f be any continuous map $R \to R$ commuting with all maps in P or in P^*. Take any a with $0 < a < 1$ and let $f(a) = \alpha$. As $\alpha = P_2 f(\sqrt{a})$, $\alpha \geqslant 0$ If $\alpha = 0$, it follows that $f(a^r) = \alpha^r = 0$ for all rational r, because $f \circ P_2 = P_2 \circ f$. Hence $f(x) = 0$ for $x \geqslant 0$; if $x \leqslant 0$, $P_2 f(x) = f(x^2) = 0$ implies again $f(x) = 0$. Thus f is identically zero.

Assume $\alpha > 0$ and let $\varepsilon \in R$ with $a^{\varepsilon} = \alpha$. Then as f and P_2 commute, $f(a^r) = a^{r\varepsilon}$ for all rational r ; hence $f(x) = x^{\varepsilon}$ for $x \geqslant 0$. If $x < 0$, then $P_2 f(x) = fP_2(x) = (x^2)^{\varepsilon}$, hence $f(x) = \pm |x|^{\varepsilon}$. As f is continuous, the lemma follows.

The situation is different for the semigroup T: this semigroup is maximal. In order to show this, we consider the following mappings of the unit interval I into itself, first introduced in [2]:

$$t_o(x) = 0 \quad \text{for all } x \;;$$

and, if $n \geqslant 1$:

$$\begin{cases} t_n(\frac{2k}{n}) = 0 \;, \quad t_n(\frac{2k+1}{n}) = 1 & (k = 0,1,2,\dots,[\frac{n}{2}]); \\ t_n \mid [\frac{k}{n}, \frac{k+1}{n}] \text{ is linear} & (k = 0,1,2,\dots, n-1). \end{cases}$$

These so-called multihats are easily seen to constitute a commutative semigroup M; in fact, $t_n \circ t_m = t_{n+m}$. In [2] P.C. Baayen, W.Kuyk and M.A. Maurice proved much more: the semigroup of all t_n, $n = 0,1,2,\dots$, <u>is a maximal</u> commutive semigroup of continuous maps $I \rightarrow I$.

<u>Lemma 2</u>. The semigroup M is equivalent to the semigroup T' of all Chebyshev polynomials T_n, restricted to the segment $[-1, +1]$, by means of the homeomorphism $\tau : [0,1] \rightarrow [-1,1]$ such that

$$\tau x = \cos \pi x \;.$$

<u>Proof</u>: immediate.

Now let f be a continuous map $R \rightarrow R$ commuting with all T_n. Then, as $f \circ T_n = T_n \circ f$ implies that f maps the set of all fixed points of T_n into itself, and as the set of all fixed points of all T_n with $n \neq 1$ is contained and dense in $[-1, +1]$, f must map this segment into itself. It then follows from lemma 2 that $f \in T$.

Hence we have shown:

<u>Lemma 3</u>. The R-cps T is maximal.

This strengthens considerably a result of G.Baxter and J.T.Joichi [3], who showed that T cannot be embedded in a 1-parameter semigroup of commuting functions.

We conclude this section with a triviality.

Lemma 4. Let Q_1, Q_2 be polynomials commuting on some non-degenerate segment. Then Q_1 and Q_2 commute everywhere on R.

3. Commutative polynomial semigroups of mappings $I \to I$

It follows from the results of section 2 that every entire I-cps is equivalent by means of a linear transformation to a semigroup $S|A$, where S is one of the R-cps T, P, P^*, and A is a closed segment $[a,b]$, $a < b$, that is invariant under S.

The only non-degenerate segment mapped into itself by T is $[-1,+1]$. The only non-trivial segments mapped into themselves by P are the segments $[-a,1]$, with $0 \leq a \leq 1$; we write P(a) for the $[-a,1]$-cps of all $P_n|[-a,1]$, $n=0,1,2,\ldots$. The only non-trivial segments invariant under P^* are the segments $[-a,b]$, with $0 \leq a \leq 1$, $a^2 \leq b \leq 1$, $b \neq 0$; we write $P^*(a,b)$ for the $[-a,b]$-cps of all $P_n|[-a,b]$, $n \geq 1$ together with $P_o^*|[-a,b]$.

Lemma 5. Each of the semigroups P(a), $0 \leq a \leq 1$, is not maximal, and is contained in a unique maximal $[-a,1]$-semigroup $\overline{P(a)}$. Similarly each $P^*(a,b)$ is contained in a unique maximal $[-a,b]$ -semigroup $\overline{P^*(a,b)}$.

Proof.

In the same way as in the proof of lemma 1 one shows that $\overline{P(a)} = \overline{P}|[-a,1]$ is the unique maximal commutative semigroup of continuous maps $[-a,1] \to [-a,1]$ containing P(a). Similarly $\overline{P^*(a,b)} = \overline{P^*}|[-a,b]$.

Hence:

Theorem 1. There are two maximal entire I-cps; they are both equivalent to T' (or to M).

Proof.

Every maximal entire I-cps must be equivalent by means of a linear map to $T' = T|[-1,+1]$. There exist two linear maps of $[-1,+1]$ onto $I = [0,1]$.

Lemma 6. If $a \neq b$, where $0 \leq a,b \leq 1$, then P(a) and P(b) are not equivalent.

Proof.

Assume P(a) and P(b) are equivalent by means of a homeomorphism $\tau : [-a,1] \to [-b,1]$. Then $\tau(1) = 1$, as 1 is the unique common fixed point of all $f \in P(a)$, and also of all $f \in P(b)$. Similarly $\tau(0) = 0$, as 0 is common fixed point of all maps but one in P(a), and also of all maps but one in P(b). Of course the second end point -a must be mapped by τ onto -b. Now it is easily seen that τ must be linear; it then follows that a = b.

The next two lemma's are proved by similar observations.

Lemma 7. Let $0 \leq a_i \leq 1$, $a_i^2 \leq b_i \leq 1$, $b_i \neq 0$ $(i=1,2)$. The semigroups $P^*(a_1,b_1)$ and $P^*(a_2,b_2)$ are equivalent if and only if $a_1 = a_2$ and $b_1 = b_2$.

Lemma 8. No semigroup P(a) is equivalent to a semigroup $P^*(b,c)$.

Consequently we have:

Theorem 2. There are infinitely many non-equivalent non-maximal entire I-cps. Each of these is equivalent by means of a linear map τ to one of the following semigroups, which are all mutually inequivalent:

$$P(a), \quad 0 \leq a \leq 1$$

or

$$P^*(a,b) \ , \ 0 \leq a \leq 1 \ , \ a^2 \leq b \leq 1, \ b \neq 0.$$

Theorem 3. Every entire I-cps is contained in a unique maximal commutative semigroup of continuous maps $I \to I$. Two entire I-cps are equivalent if and only if the maximal commutative semigroups in which they are contained are equivalent.

4. Remark on mappings commuting with T_n or P_n, $n \geq 2$.

It was shown by P.C. Baayen and W. Kuyk in [1] that every open map of I into itself that commutes with t_2 is itself a multihat t_n. From this it follows almost at once that every continuous map commuting with t_2 is either a t_n or is everywhere oscillating (nowhere monotone).

This result has been improved very much by G.Baxter and J.T. Joichi [4], who showed the following theorem:

If a continuous map $f : I \to I$ commutes with some multihat t_n, $n \geq 2$, it is itself either a hat-function or a constant map.

Now we saw in section 2 that the semigroup M of all hats t_n is equivalent to the semigroup T' of all Chebyshev polynomials on $[-1,+1]$.
Hence we conclude:

Theorem 4. Every non-constant continuous map of $[-1,+1]$ into itself that commutes with a Chebyshev polynomial T_n with $n \geq 2$, is itself a Chebyshev polynomial.

For the maps P_n, $n \geq 2$, the situation is completely different. Consider e.g. continuous maps of $[0,1]$ into itself which commute with P_2 on that interval.
There exist multitudes of such functions. For let $0 < a < 1$, and let f_0 be any continuous function of (a^2,a) into $(0,1)$. If we define: $f(0)=0$, $f(1)=1$, $f(x)=(f_0(x^{2^{-n}}))^{2^{-n}}$ if $x \in (a^{2^n},a^{2^{n-1}})$ (n integer), f will be a continuous map $I \to I$ commuting with P_2.

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